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Fermions from half-BPS supergravity

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Abstract: We discuss collective coordinate quantization of the half-BPS geometries of Lin, Lunin and Maldacena [1]. The LLM geometries are parameterized by a single function u on a plane. We treat this function as a collective coordinate. We arrive at the collective coordinate action as well as path integral measure by considering D3 branes in an arbitrary LLM geometry. The resulting functional integral is shown, using known methods ([2]), to be the classical limit of a functional integral for free fermions in a harmonic oscillator. The function u gets identified with the classical limit of the Wigner phase space distribution of the fermion theory which satisfies $u * u = u$. The calculation shows how configuration space of supergravity becomes a phase space (hence noncommutative) in the half-BPS sector. Our method sheds new light on counting supersymmetric configurations in supergravity.

Keywords: AdS-CFT and dS-CFT Correspondence, D-branes.

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1. Introduction

Recently it has been shown in [1] that the half-BPS IIB supergravity solutions, which are asymptotically $AdS_5 \times S^5$ and preserve an $\mathrm{O}(4) \times \mathrm{O}(4)$ symmetry of the asymptotic isometry group, are in one-to-one correspondence with semiclassical configurations of free fermions in a harmonic oscillator potential. This result is yet another striking evidence of the AdS/CFT correspondence [3], since the free fermions are equivalent to [4] the half-BPS sector of the super Yang-Mills theory. Related work can be found in [5, 4−18].

The correspondence between the supergravity configurations and semiclassical fermion configurations is based on a proposed identification between a supergravity mode $u(x_1, x_2)$ with the phase space density $u(q, p)$ of the free fermions, where x_1, x_2 are two of the coordinates of the LLM geometry and q, p are coordinates of the phase space of the free fermions. The present work began with the questions (a) how two coordinates of space time can become phase space (noncommutative) coordinates and (b) whether one can derive the noncommutative dynamics directly from supergravity.

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The plan of the paper is as follows. In section 2, we mention a few results of [1] to identify the moduli space of half-BPS vacua. The moduli space is parameterized by a single function $u(x_1, x_2)$ (discussed in the previous paragraph) subject to two constraints. In section 3 we quantize the half-BPS configurations by identifying u as the collective coordinate. We provide a parameterization of the generic function u subject to the constraints and identify them with D3 branes coupled to LLM geometries. The collective coordinate actions are then calculated by computing the D3 brane actions. We use the formalism of phase space path integrals to demonstrate how the phase space dimensions get reduced by half under the BPS constraint and the configuration space itself becomes a phase space. In section 4 we collect the results and rewrite the action as well as the measure in terms of the u-variable. In section 5 we identify the u-functional integral with the classical limit of a functional integral describing free fermions in a harmonic oscillator. In section 6 we discuss a first principles approach to derivation of the u-functional integral using the general formalism of collective coordinates in the presence of BPS constraints, using Kirillov's symplectic form. Section 7 contains a summary and some open questions. In appendix A we present some details concerning identification of the collective coordinate action of section 4 with the D3-brane actions of section 3. Appendix B makes a qualitative identification between gravitons and collective excitations in the form of ripples.

Transformation of configuration space into a phase space under BPS conditions has been considered in [19] in the case of a giant graviton probe in $AdS_5 \times S^5$. Supertubes have been discussed in somewhat related contexts in [20, 21]. Rather appealing similarities with parts of the present work can be found in discussions on topological string/field theories [22−24]. Related ideas have also appeared in the context of quantum hall systems in [25, 26].

2. The moduli space of 1/2-BPS Supergravity

As shown in [1], the half-BPS geometries (with $\mathrm{O}(4)\times\mathrm{O}(4)$ symmetry) are characterized by a single function $z_0(x_1, x_2) \equiv z(x_1, x_2, y = 0)$ (see [1, eqs. (2.5)–(2.15)]). The moduli space of these solutions is the space of z_0's, subject to the following regularity and topological constraints.

The regularity constraint. The constraint of regularity on the half-BPS geometries implies that z_0 can only be either 1/2 or −1/2, that is [1]

$$z_0(x_1, x_2) = -\frac{1}{2}\sum_i \chi_{R_i} + \frac{1}{2}\sum_j \chi_{\tilde{R}_j}\,, \tag{2.2}$$

where the x_1, x_2 plane is tessellated by the regions $R_i, \tilde{R}_j$, with $z_0 = -1/2$ in R_i and $z_0 = 1/2$ in the $\tilde{R}_j$.

[1] $\chi_R(x)$ denotes the characteristic function of a region $R \subset \mathbb{R}^2$:

$$\chi_R(x) = 1 \text{ if } x \in R\,, = 0 \text{ otherwise}\,. \tag{2.1}$$

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It is useful to define the function

$$u(x_1, x_2) \equiv \frac{1}{2} - z_0(x_1, x_2)\,. \tag{2.3}$$

The regularity constraint now reads $u(x_1, x_2) = 0$ or 1, equivalently[2]

$$(u(x_1, x_2))^2 = u(x_1, x_2)\,. \tag{2.4}$$

The equation (2.2) becomes

$$u = \sum_i \chi_{R_i}(x_1, x_2)\,, \tag{2.5}$$

where R_i now denote regions with $u = 1$.

The topological constraint. The topological constraint becomes [1]

$$\begin{aligned} \int_{R_i} \frac{dx_1 dx_2}{2\pi\hbar} &= N_i \\ \int_{-\infty}^{\infty} \frac{dx_1 dx_2}{2\pi\hbar} u &= \sum_i N_i = N\,, \end{aligned} \tag{2.6}$$

where

$$\hbar = 2\pi g_s \alpha'^2\,. \tag{2.7}$$

The condition that the geometries are asymptotically $AdS_5 \times S^5$ implies that $R = \cup R_i$ is a bounded region of the x_1, x_2 plane.

The functions $u(x_1, x_2)$ subject to the constraint equations (2.4) and (2.6) characterize all regular half-BPS solutions of the system with $\mathrm{O}(4) \times \mathrm{O}(4)$ symmetry and $AdS_5 \times S^5$ asymptotics.

3. Quantization of half-BPS vacua

We will treat the function u as the collective coordinate of the space of half-BPS configurations (with $\mathrm{O}(4) \times \mathrm{O}(4)$ symmetry). The space of u's can be discussed in terms of orbits of a specific u_0 under the action of the group of area-preserving diffeomorphisms in two dimensions (see section 6 for this description). Alternatively, u can be parameterized as in (2.5). By choosing generic enough regions R_i, we can describe all functions u subject to the constraints. This is the description we will use in this and the following two sections to quantize the space of u's.

Let us choose the regions as follows (see figure 1):

$$u(x_1, x_2) = u_0(x_1, x_2) - \sum_{j=1}^{m} \chi_{H_j}(x_1, x_2) + \sum_{i=1}^{n} \chi_{P_i}(x_1, x_2)\,. \tag{3.1}$$

[2] $0 < u < 1$ gives rise to singular solutions; e.g. [27] identified the superstar solution [28–30] with $0 < u < 1$. [27] also showed in specific examples that the geometries with $u > 1$ develop closed timelike curves.

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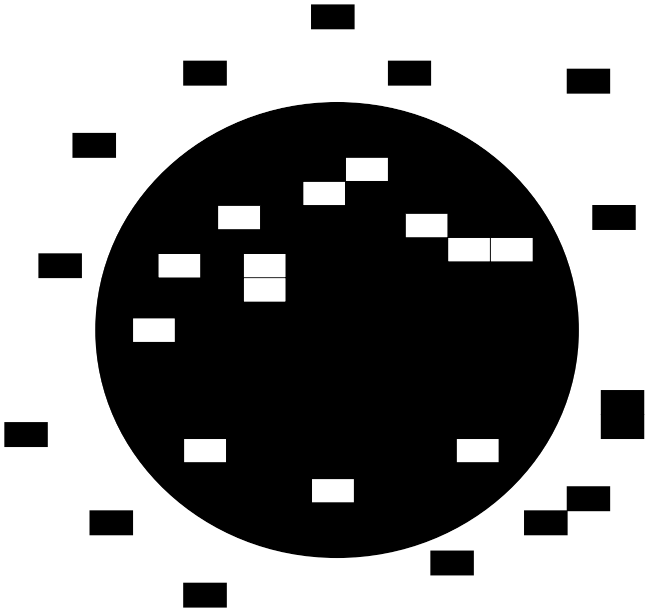

Figure 1: Checkerboard parameterization. The white rectangles inside the circle represent the regions H_j in (3.1), while the black rectangles outside the circle denote the regions P_i. A small number of isolated cells represents giant gravitons in S^5 or in AdS_5. When the number of cells is large, each additional cell (black or white) can be regarded as a D3 brane in an arbitrary background LLM geometry defined by the rest of the pattern.

Here u_0 represents a filled circle of radius r_0:

$$u_0 = \theta(r_0 - r) \tag{3.2}$$

and the regions H_j, P_i are non-intersecting rectangular cells, with H's (holes) inside the circle of radius r_0 and P's outside the circle.

The constraint (2.4) is obviously satisfied. The other constraint (2.6) can also be easily satisfied, by choosing the area of each of the cells H_j or P_i to be integral (in units of $2\pi\hbar$) and by choosing the radius r_0 in (3.2) so as to keep the total area equal to N. Clearly the minimum area of the cells H_j or P_i is $2\pi\hbar$. In the limit of a large number of such cells, arbitrarily scattered, we can recover a rather general[3] representation of the type (2.5), subject to (2.4) and (2.6).

Thus, in (3.1) we will choose the H_j to be minimum area cells (we will take them to be squares without loss of generality, with each side equal to $\sqrt{2\pi\hbar} \equiv \epsilon$), with centres denoted by $(x_1^j, x_2^j), j = 1, \ldots, m$. Similarly we will take P_i's to be squares of the same minimal size, with centres denoted by $(x_1^i, x_2^i), i = 1, \ldots, n$.

The specific rectangular shape of the cells is not important for our discussions (except for visualizing a simple tiling[4]). The same results could be derived, e.g. by using cells with sides along the r and ϕ directions.

[3] See footnote 4.

[4] The tiling is only in an approximate sense since we will regard the cell boundaries as separated by distances $> \mathrm{O}(\sqrt{\hbar})$ to prevent high curvatures arising from droplets that are too close; such inter-cell

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3.1 Correspondence between checkerboard configurations and IIB geometries

The correspondence with IIB geometries, following [1], is described below:

(a) When there are no H's or P's, the circle of radius r_0 represents $AdS_5 \times S^5$, where r_0 is given by (3.7).

(b) A configuration (3.1) with a small number of non-intersecting minimum-area cells P_i and H_j represents giant gravitons wrapping the three-spheres of AdS_5 or S^5, so that the background configuration is essentially the same as in the case (a) above. The cell P_i will represent the i-th giant graviton extending in AdS_5 (such giant gravitons are called "dual giant gravitons" [31, 32]). The centre of mass of the giant graviton will be identified with the centre (x_1^i, x_2^i) of the cell P_i. Similarly, the cell H_j will represent the j-th giant graviton extending in S^5 [33]. The centre of mass of the giant graviton will be identified with the centre (x_1^j, x_2^j) of the cell H_j.

(c) A single minimum-area cell H_j (hole) inside the filled part of a generic u-configuration (representing an arbitrary LLM geometry) will be identified as a D3-brane wrapping the three-sphere $\tilde{S}^3$ of that geometry[5] (see (3.25)).

(d) Similarly, a minimum area cell P_i in the unfilled part of an arbitrary u-configuration will be identified as a D3-brane wrapping the three-sphere S^3 of the corresponding geometry (see (3.39)).

3.2 Recipe for the collective coordinate action

We will derive the collective coordinate action[6] based on the above correspondences. For example, for configurations (b), the collective coordinate action for the u-fluctuation represented by a cell H_j or P_i will be identified with the action of the corresponding giant (or dual giant) graviton, subject to the half-BPS constraint. Similarly, for configurations (c) and (d), the collective coordinate action will be identified with the action of the corresponding D3-branes in an arbitrary LLM geometry, subject to the half-BPS conditions.

To describe our method, let us consider the example of the case (c), where we create a 'hole' (a white pixel) at the position $(\bar{x}_1, \bar{x}_2)$. This changes the initial u-configuration from an arbitrary initial value u_0 to $u_0 - \delta u$ (where δu is given by (3.4) for a rectangular hole). As mentioned above, this deformation δu should be identified with a BPS D3 brane which wraps the 3-sphere $\tilde{S}^3$ of the LLM geometry u_0 and is located at $(\bar{x}_1, \bar{x}_2)$. The collective

separations can be interpreted in terms of fuzzy u-configurations satisfying (5.5) in the finite $\hbar$ theory (see sections 5 and 6). Droplets closer than this distance can be assumed to merge, leading to "ripples". These are proposed in [1] to correspond to gravitons; we briefly explore the correspondence between gravitons and the collective action for ripples in appendix B.

[5] We will not consider collective excitations corresponding to multiple D3 branes, except to remark that two D3 branes which are classically on top of each other are described in [1] as a spread-out u configuration occupying twice the area, to be consistent with the constraints (2.4) and (2.6). This accords with the fermionic description (5.1) or (5.8) which we ultimately arrive at.

[6] An independent derivation, more directly from supergravity, based on Kirillov's symplectic form, is briefly sketched in section 6 (see point (2) and the references therein for details.)

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coordinate action $S[u]$ that we are looking for should, therefore, satisfy the property that $\delta S = S[u_0 - \delta u] - S[u_0]$ should be identical to the action S_{D3}^{BPS} (DBI + CS) of the above-mentioned D3 brane.[7]

Similar considerations apply to the case (d), where one adds a 'particle' (a black pixel) at the position $(\bar{x}_1, \bar{x}_2)$ so that $u_0 \to u_0 + \delta u$, with δu given by (3.4). In this case one demands that $S[u]$ should satisfy the property that the change in $S[u]$ should be equal to the action of a half-BPS D3-brane at that position, wrapping S^3. The case (b) is of course simpler where the background geometry is $AdS_5 \times S^5$ and the D3-branes are the usual giant or dual giant gravitons.

With the above understanding of terms, the classical action $S[u]$ should satisfy the property

$$\delta S = S_{\rm D3}^{\rm BPS} \tag{3.3}$$

for an arbitrary choice of the fluctuation $\pm\delta u$, around any background u_0.

We will find that such an $S[u]$ indeed exists (same as the one obtained using the Kirillov form, section 6).

Besides a classical action $S[u]$, we will also find a measure $D[u]$ such that the measure for the fluctuation $D[\delta u]_{\tilde{u}_0}$ agrees with the path integral measure of the D3-brane dynamics.

Note that we are making the identification of the D3 brane degrees of freedom with the collective coordinates of the supergravity background. We are assuming this, as in [1]. This is similar in spirit with the original identification by Polchinski [34, 35] of Dirichlet branes as collective coordinates of supergravity backgrounds carrying Ramond-Ramond charges.

We will discuss a more first principles approach in a later section (section 6).

Let us now consider, in turn, the D3-branes corresponding to configurations (b), (c) and (d) of section 3.1.

3.3 Single giant graviton in $AdS_5 \times S^5$

In this and the next subsections we will describe the calculation of the right hand side of (3.3) in the cases (b), (c) and (d) respectively. In section 4 the action $S[u]$ and the calculation of δS in the left hand side of (3.3) will be discussed.

We will first consider a giant graviton extending in S^5 [33]. As discussed above, this corresponds to a hole H with each side equal to $\epsilon = \sqrt{2\pi\hbar}$. We will denote the centre of H as $(\bar{x}_1, \bar{x}_2)$. The change in the u-function corresponding to creation of the hole is $-\delta u$ where

$$\begin{aligned} \delta u &= \chi_{\bar{x}_1,\bar{x}_2}(x_1, x_2) \\ &\equiv \theta\left(\bar{x}_1 + \frac{\epsilon}{2} - x_1\right)\theta\left(-\bar{x}_1 + \frac{\epsilon}{2} + x_1\right)\theta\left(\bar{x}_2 + \frac{\epsilon}{2} - x_2\right)\theta\left(-\bar{x}_2 + \frac{\epsilon}{2} + x_2\right). \end{aligned} \tag{3.4}$$

[7] Note that the configuration $u_0 - \delta u$ does not preserve the area constraint (2.6). So we must create another deformation $+\delta u'$ by adding a "particle", or inflating the periphery of one of the droplets comprising u_0. In principle δS could depend on the choice of $+\delta u'$; however, it is easy to show that the effect is subleading in $1/N$ and we will ignore it. This is consistent with the fact that the action $S[u]$ we will arrive at agrees with the fermion action in the semiclassical limit. We will discuss this further in section 7, point (6).

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We will now discuss the calculation of the right hand side of (3.3), namely the giant graviton action. Half-BPS configurations of a giant graviton extending in S^5 of $AdS_5 \times S^5$have been discussed in [19]. The giant graviton is a D3-brane with the embedding (in static gauge)

$$t = \tau\,, \qquad \theta = \theta(\tau)\,, \qquad \tilde{\phi} = \tilde{\phi}(\tau)\,, \qquad \tilde{\Omega}_i = \sigma_i\,, \qquad \rho = 0 \tag{3.5}$$

where we have used global coordinates of $AdS_5 \times S^5$, defined by the metric

$$ds^2 = r_0 \left[-\cosh^2\!\rho\, dt^2 + d\rho^2 + \sinh^2\!\rho\, d\Omega_3^2 + \cos^2\!\theta\, d\tilde{\phi}^2 + d\theta^2 + \sin^2\!\theta\, d\tilde{\Omega}_3^2 \right]. \tag{3.6}$$

Here

$$r_0^2 = R_{\rm AdS}^4 = 4\pi N l_p^4 = 4\pi N g_s \alpha'^2\,. \tag{3.7}$$

The relation to the LLM coordinates is

$$\begin{aligned} r &= r_0 \cosh\rho \cos\theta \\ y &= r_0 \sinh\rho \sin\theta \end{aligned} \tag{3.8}$$

and

$$\phi = \tilde{\phi} + t\,. \tag{3.9}$$

For $y = 0$, we have

$$r = r_0 \cos\theta\,. \tag{3.10}$$

We have used the notation (r, ϕ) as polar coordinates for the (x_1, x_2) plane. The D3 brane action is given by[8]

$$S = N \int d\tau \left[-\sin^3\theta \sqrt{1 - \cos^2\theta\, \dot{\tilde{\phi}}^2 - \dot{\theta}^2} - \sin^4\theta\, \dot{\tilde{\phi}} \right]. \tag{3.11}$$

The factor N in front arises as

$$N = T_3 \omega_3 r_0^2\,, \tag{3.12}$$

where $T_3 = 1/(8\pi^3\alpha'^2 g_s)$ is the D3-brane tension, $\omega_3 = 2\pi^2$ is the volume of the unit S^3 and r_0^2 is given in (3.7).

The configuration space of the giant graviton is given by $\theta(\tau), \tilde{\phi}(\tau)$. This corresponds to a four-dimensional phase space $\theta(\tau), p_\theta(\tau), \tilde{\phi}(\tau), p_{\tilde{\phi}}(\tau)$. It is easy to see that for BPS configurations we must have [19]

$$\dot{\theta} = 0\,, \qquad \dot{\tilde{\phi}} = -1 \tag{3.13}$$

or, equivalently,

$$p_\theta = 0\,, \qquad p_{\tilde{\phi}} = -N \sin^2\theta\,. \tag{3.14}$$

[8] $\tilde{\phi}$ here is $-\phi$ of [19].

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In [19] the BPS constraints (3.14) were imposed as Dirac constraints on the four dimensional phase space. The result was a two dimensional phase space which could be coordinatized by $\theta, \tilde{\phi}$ which satisfied the following Dirac bracket:

$$\{\theta, \tilde{\phi}\}_{\rm DB} = \frac{1}{2N \sin\theta \cos\theta}\,, \qquad \text{or} \qquad \{\sin^2\theta, \phi\}_{\rm DB} = \frac{1}{N}\,. \tag{3.15}$$

The hamiltonian in the reduced phase space is given by [9]

$$\tilde{H} = -p_{\tilde{\phi}} = N \sin^2\theta\,. \tag{3.16}$$

Another way of stating the above result is that the unconstrained path integral for the system

$$Z_{\rm full} = \int D\theta(\tau) Dp_\theta(\tau) D\tilde{\phi}(\tau) Dp_{\tilde{\phi}}(\tau) \exp\left[i \int d\tau \left(\dot{\tilde{\phi}} p_{\tilde{\phi}} + \dot{\theta} p_\theta - H_{\rm full}\right)\right] \tag{3.17}$$

reduces, under the BPS constraints, to the following path integral

$$Z_{\rm BPS} = \int D[\sin^2\theta(\tau)] D[\tilde{\phi}(\tau)] \exp\left[i \int d\tau \left(-N \sin^2\theta \dot{\tilde{\phi}} - \tilde{H}\right)\right] \tag{3.18}$$

where $\tilde{H}$ is given by (3.16). We will show in section 4 how the above functional integral can be written in terms of the u-variable in the sense of Section 3.2 (in particular (3.3)).

The treatment of the dual giant graviton, extending into AdS_5 [31, 32] of $AdS_5 \times S^5$, is very similar. The corresponding u-configuration consists of a single cell P_i outside of u_0. Thus,

$$u(x_1, x_2) = u_0 + \delta u\,, \tag{3.19}$$

where δu is again given by the expression in (3.4).

The D3 brane embedding for the dual giant graviton is

$$t = \tau\,, \qquad \rho = \rho(\tau)\,, \qquad \tilde{\phi} = \tilde{\phi}(\tau)\,, \qquad \Omega_i = \sigma_i\,, \qquad \theta = 0\,. \tag{3.20}$$

For this embedding, r gets related to ρ as follows:

$$r = r_0 \cosh\rho\,. \tag{3.21}$$

The BPS constraints are:

$$p_\rho = 0\,, \qquad p_{\tilde{\phi}} = -N \sinh^2\rho\,. \tag{3.22}$$

The constrained path integral (the analog of (3.18)) now is

$$\begin{aligned} Z_{\rm BPS} &= \int D[\sinh^2\rho(\tau)] D[\tilde{\phi}(\tau)] \exp\left[i \int d\tau \left(-N \sinh^2\rho\, \dot{\tilde{\phi}} - \tilde{H}\right)\right] \\ \tilde{H} &= -p_{\tilde{\phi}} = N \sinh^2\rho\,. \end{aligned} \tag{3.23}$$

We will show in section 4 that this is also a special case of the same u-path integral as the earlier example was.

[9] If we use the "moving coordinate" ϕ, the hamiltonian becomes $H = \tilde{H} + p_{\tilde{\phi}} = \tilde{H} + p_\phi = 0$. This is a reflection of the relation $\partial/\partial t|_\phi = \partial/\partial t|_{\tilde{\phi}} + \partial/\partial\tilde{\phi}|_t$. See also remarks below equation (4.2).

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3.4 D3 brane in arbitrary LLM geometry

Let us first consider configuration (c) of section 3.1, where we have a single cell H (hole) inside a filled (black) region of an arbitrary u-configuration, which we will write as (see section 3.2)

$$u(x_1, x_2) = u_0 - \delta u\,, \tag{3.24}$$

where δu is again as in (3.4), but u_0 represents an arbitrary background u-configuration, satisfying the constraints (2.6), (2.4). We will ignore here the area-compensating change $\delta u'$ as discussed in footnote 7.

The D3 brane corresponding to the fluctuation (3.24) is described by the following embedding (using the LLM coordinates, see (3.26)):

$$t = \tau\,, \qquad x_1 = \bar{x}_1(\tau)\,, \qquad x_2 = \bar{x}_2(\tau)\,, \qquad y = 0\,, \qquad \tilde{\Omega}_m = \sigma_m\,, \qquad m = 1, 2, 3\,. \tag{3.25}$$

Let us discuss the geometry corresponding to u_0. Recall that the LLM metric is of the form [1]

$$ds^2 = g_{tt}(dt + V_i dx_i)^2 + g_{yy}(dx_i dx_i + dy^2) + g_{\Omega\Omega}\; d\Omega_3^2 + g_{\tilde{\Omega}\tilde{\Omega}}\; d\tilde{\Omega}_3^2\,, \tag{3.26}$$

where $d\Omega_3^2, d\tilde{\Omega}_3^2$ represent metric on two unit 3-spheres S^3 and $\tilde{S}^3$ respectively (the two 3-spheres are distinguished by the fact that S^3 has vanishing radius in the $u = 1$ region of the $\vec{x}$-plane (see (3.27)), whereas $\tilde{S}^3$ has vanishing radius in the $u = 0$ region of the $\vec{x}$-plane (see (3.37)). The parts of the metric and RR background which are important for us are near $y = 0$:

$$\begin{aligned}
u_0 &= 1 - y^2 f \\
V_i &= v_i\,, \qquad i = 1, 2 \\
-g_{tt} &= \frac{1}{g_{yy}} = f^{-1/2} \\
g_{\Omega\Omega} &= y^2\sqrt{f} \\
g_{\tilde{\Omega}\tilde{\Omega}} &= f^{-1/2} \\
B_t &= -\frac{1}{4} y^4 f \\
\tilde{B}_t &= -\frac{1}{4f} \\
d\hat{B} &= -\frac{1}{4} y^3 *_3 df \\
d\tilde{\hat{B}} &= -\frac{1}{2} dx_1 \wedge dx_2 = -\frac{1}{4} d(x_1\; dx_2 - x_2\; dx_1)\,.
\end{aligned} \tag{3.27}$$

Here $*_3$ is the flat space epsilon symbol in the three dimensions parameterized by y, x_1, x_2. All expressions on the right hand sides are understood to be multiplied by $(1 + O(y^2))$. $f(x_1, x_2), v_i(x_1, x_2)$ are both obtainable from $u_0(x_1, x_2)$. Explicitly,

$$\begin{aligned}
f(\vec{x}) &= \mathrm{Limit}_{y \to 0} \left[\frac{1}{y^2} - \frac{1}{\pi} \int_D \frac{d^2\vec{x}'}{[(\vec{x} - \vec{x}')^2 + y^2]^2} \right] \\
v_i(\vec{x}) &= \frac{\epsilon_{ij}}{2\pi} \oint_{\partial D} \frac{dx'_j}{(\vec{x} - \vec{x}')^2}\,.
\end{aligned} \tag{3.28}$$

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Here D denotes the support of u. The limit for f is well-defined since the explicit $1/y^2$ cancels with a $1/y^2$ coming from the $\vec{x} = \vec{x}'$ region of the integral. It is easy to calculate explicit forms for f, for example, for ring configurations of $\tilde{u}_0$.

Under the approximations (3.27) the metric and the RR 4-form field are given, upto $(1 + O(y^2))$, by

$$\begin{aligned} ds^2 &= \frac{[-(dt + v_i dx_i)^2 + f(dx_1^2 + dx_2^2 + d\vec{y}^2) + d\tilde{\Omega}^2]}{\sqrt{f}} \\ C^{(4)} &= -\frac{1}{4}\left[\frac{dt + v_i dx_i}{f} + r^2 d\phi\right] \wedge d^3\tilde{\Omega} \end{aligned} \tag{3.29}$$

where $d\vec{y}^2 = dy^2 + y^2 d\Omega^2$, and $d^3\tilde{\Omega}$ is the volume form of the three-sphere $\tilde{S}^3$ (see (3.26)).

The D3 brane action is given by[10] (dropping the bar's on $x_i(t)$ in (3.25))

$$\begin{aligned} S &= T_3\omega_3 \int d\tau \left[-\frac{1}{f}\sqrt{(1 + v_r\dot{r} + v_\phi\dot{\phi})^2 - f(\dot{r}^2 + r^2\dot{\phi}^2)} + r^2\dot{\phi} + \frac{1}{f}(1 + v_r\dot{r} + v_\phi\dot{\phi})\right] \\ &= \frac{1}{2\hbar}\int dt \bigg[-\frac{1}{f}\sqrt{(1 + v_r\dot{r} + v_\phi(\dot{\tilde{\phi}} + 1))^2 - f(\dot{r}^2 + r^2(\dot{\tilde{\phi}} + 1)^2)} + \\ &\qquad + r^2(\dot{\tilde{\phi}} + 1) + \frac{1}{f}(1 + v_r\dot{r} + v_\phi(\dot{\tilde{\phi}} + 1))\bigg]\,. \end{aligned} \tag{3.30}$$

The BPS conditions can be obtained by the constraint $\tilde{H} = -p_{\tilde{\phi}}$, which gives

$$\dot{\tilde{\phi}} = -1\,, \qquad \dot{r} = 0\,. \tag{3.31}$$

In the ϕ, t coordinates

$$\dot{\phi} = 0\,, \qquad \dot{r} = 0\,. \tag{3.32}$$

The hamiltonian H in the LLM frame is $H = 0$ (see footnote 9). It should be possible to derive these equations from an analysis of the Killing spinor and world-volume kappa-symmetry, but another way of seeing the validity of equations (3.32) is that it is equivalent to time-independence of δu in (3.24). Any such time-independent u-configuration is half-BPS, as shown in [1]; indeed the half-BPS condition does not allow any time-dependence of u. Hence (3.32) is equivalent to the Killing spinor condition.

The remaining analysis is similar to the case of the giant gravitons in $AdS_5 \times S^5$. On the constrained surface (3.32) we have

$$p_r = 0\,, \qquad p_{\tilde{\phi}} = \frac{1}{2\hbar}r^2\,. \tag{3.33}$$

The hamiltonian is given by

$$\tilde{H} = -p_{\tilde{\phi}} = -\frac{1}{2\hbar}r^2\,, \tag{3.34}$$

the negative sign reflecting the energy of a hole.

[10] Note the appearance in the second line of the $\hbar$ of (2.6),(2.7) through the equality $T_3 w_3 = N/r_0^2 \equiv 1/(2\hbar)$, cf. (3.12).

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The constrained path integral, the analog of (3.18), now becomes

$$\begin{aligned} Z_{\rm BPS} &= \int D[r^2(\tau)] D[\tilde\phi(\tau)] \exp[iS_{\rm BPS}] \\ S_{\rm BPS} &= \int d\tau \left(\frac{1}{2\hbar} r^2 \dot{\tilde\phi} - \tilde H \right), \end{aligned} \tag{3.35}$$

where $\tilde H$ is given by (3.34). To compare with (3.18), note that on (3.10) $r^2/(2\hbar) = N\cos^2\theta = N - N\sin^2\theta$. The extra N is explained in the paragraph following (4.5).

Let us now consider configuration (d), where we have a single (black) cell P in a white region of an arbitrary u-configuration. The full u-configuration, including contribution from P is given by

$$u(x_1, x_2) = u_0 + \delta u\,, \tag{3.36}$$

where δu is given by (3.4).

As in (3.27), the important parts of the metric and RR background are near $y = 0$. These are now given by

$$\begin{aligned} u_0 &= y^2 f \\ V_i &= v_i \\ -g_{tt} &= \frac{1}{g_{yy}} = f^{-1/2} \\ g_{\Omega\Omega} &= f^{-1/2} \\ g_{\tilde\Omega\tilde\Omega} &= y^2\sqrt{f} \\ B_t &= -\frac{1}{4f} \\ \tilde B_t &= -\frac{1}{4} y^4 f \\ d\hat B &= \frac{1}{2} dx_1 \wedge dx_2 = \frac{1}{4} d(x_1\ dx_2 - x_2\ dx_1) \\ dB &= \frac{1}{4} y^3 * df\,. \end{aligned} \tag{3.37}$$

All expressions on the right hand sides are understood to be multiplied by $(1 + O(y^2))$. v_i are again given by (3.28), while $f = (1/\pi) \int_D d^2\vec x'\ (\vec x - \vec x')^{-4}$.

The metric and the RR form are given by

$$\begin{aligned} ds^2 &= \frac{[-(dt + v_i dx_i)^2 + f(dx_1^2 + dx_2^2 + d\vec y^2) + d\Omega^2]}{\sqrt{f}} \\ C^{(4)} &= -\frac{1}{4}\left[\frac{dt + v_i dx_i}{f} - r^2 d\phi\right] \wedge d^3\Omega\,, \end{aligned} \tag{3.38}$$

where $d\vec y^2 = dy^2 + y^2 d\tilde\Omega^2$ and $d^3\Omega$ represents the volume-form on S^3 (see (3.26)).

Let us consider the D3 brane represented by δu in (3.36). Its embedding is given by (using, again, the LLM coordinates of (3.26))

$$t = \tau\,, \qquad x_1 = \bar x_1(\tau)\,, \qquad x_2 = \bar x_2(\tau)\,, \qquad y = 0\,, \qquad \Omega_m = \sigma_m\,, \qquad m = 1, 2, 3\,. \tag{3.39}$$

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The D3 brane action, analogous to (3.30), is given by (dropping the bar's on $x_i(\tau)$)

$$S = T_3\omega_3 \int d\tau \left[-\frac{1}{f}\sqrt{(1+v_r\dot r+v_\phi\dot\phi)^2 - f(\dot r^2+r^2\dot\phi^2)} - r^2\dot\phi + \frac{1}{f}(1+v_r\dot r+v_\phi\dot\phi)\right]$$
$$= \frac{1}{2\hbar}\int d\tau \Bigg[-\frac{1}{f}\sqrt{(1+v_r\dot r+v_\phi(\dot{\tilde\phi}+1))^2 - f(\dot r^2+r^2(\dot{\tilde\phi}+1)^2)} -$$
$$- r^2(\dot{\tilde\phi}+1) + \frac{1}{f}(1+v_r\dot r+v_\phi(\dot{\tilde\phi}+1))\Bigg]. \tag{3.40}$$

The BPS condition $H = -p_{\tilde\phi}$, once again equivalent to $\dot{\tilde\phi} = -1, \dot r = 0$, implies that the BPS dynamics is described by the path integral (analog of (3.35))

$$\begin{aligned} Z_{\rm BPS} &= \int D[r^2(\tau)]D[\tilde\phi(\tau)]\exp[iS_{\rm BPS}] \\ S_{\rm BPS} &= \int d\tau\left(-\frac{r^2}{2\hbar}\dot{\tilde\phi} - \tilde H\right) \\ \tilde H &= -p_\phi = \frac{1}{2\hbar}r^2 . \end{aligned} \tag{3.41}$$

Note that the hamiltonian for the filled cell is positive this time. For comparison with (3.23), remarks similar to the ones below (3.35) apply here as well (note that according to (3.21) $r^2/(2\hbar) = N\cosh^2\rho = N + N\sinh^2\rho$).

4. Collective coordinate action

We will now show that all the path integrals (3.18), (3.23), (3.35) and (3.41) are equivalent to the following path integral in terms of the u-variable:

$$\begin{aligned} Z &= \int Du\exp[iS_{\rm BPS}] \\ S_{\rm BPS} &= \int \frac{dx_1dx_2}{2\pi\hbar}\hbar\int_{\tilde\Sigma} d\tau\ ds\ u(x_1,x_2,\tau,s)\{\partial_\tau u,\partial_s u\}_{\rm PB} - \int_\Sigma d\tau\tilde H \\ \tilde H &= \int\frac{dx_1dx_2}{2\pi\hbar}u(x_1,x_2,\tau)\frac{x_1^2+x_2^2}{2\hbar} \end{aligned} \tag{4.1}$$

Here Σ denotes a curve $\tau\mapsto u(x_1,x_2,\tau)$ in the u-configuration space and $\tilde\Sigma$ denotes a one-parameter extension of Σ to the map $(\tau,s)\mapsto u(x_1,x_2,\tau,s), s<s_0$, such that $u(x_1,x_2,\tau,s_0) = u(x_1,x_2,\tau)$. Although in order to write the action we need to introduce the s-extension, it can be easily shown that the extension does not affect the path integral as long as the boundary value (at $s=s_0$) remains $u(x_1,x_2,\tau)$ (this follows from the fact that the symplectic form appearing in (4.1) is closed). In this and the following section we use

$$(x_1,x_2) = (r\cos\tilde\phi, r\sin\tilde\phi) \tag{4.2}$$

(see eq. (3.9)). The $\tilde\phi$ coordinate, rather than ϕ, is the more natural angle to use for comparison with the boundary theory, because, e.g. the time-derivative in the boundary theory is the operator $\partial/\partial t|_{\tilde\phi}$ appearing in footnote 9. In terms of (r,ϕ) coordinates the hamiltonian is zero (see footnote 9).

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The notation $\{\}_{\rm PB}$ is defined here as

$$\{f,g\}_{\rm PB} \equiv \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial g}{\partial x_1}\frac{\partial f}{\partial x_2}\,.$$

We will see later (see section 6 and references therein) that the action (4.1) is Kirillov's coadjoint orbit action for the group of area-preserving diffeomorphisms.

The measure Du, described in sections 4.2 and 6, incorporates the constraints (2.4) and (2.6). The equation of motion for $u(x_1,x_2,\tau)$ that follows from (4.1) is (see [36, 37]):

$$\partial_\tau u - (x_1\partial_2 - x_2\partial_1)u = 0\,. \tag{4.3}$$

4.1 Action

We will show that the action (4.1) gives rise to the various D3-brane actions in (3.18), (3.23), (3.35) and (3.41) in the sense of (3.3). Consider, for example, configuration (d), (3.36), (3.39). It is easy to see that if δu does not intersect with u_0, then the left hand side of (3.3) is given by local properties of the cell δu, viz. $\delta S[u] = S[\delta u]$. Thus we get

$$\begin{aligned}
\delta S &= \delta S_{\rm kin} - \delta S_{\rm ham} \\
\delta S_{\rm kin} &= \int \frac{dx_1 dx_2}{2\pi\hbar}\hbar \int_{\tilde\Sigma} d\tau ds\ \delta u\{\partial_\tau \delta u, \partial_s \delta u\}_{\rm PB} \\
\delta S_{\rm ham} &= \int d\tau \int \frac{dx_1 dx_2}{2\pi\hbar}\delta u(x_1,x_2,\tau)\frac{x_1^2+x_2^2}{2\hbar}\,.
\end{aligned} \tag{4.4}$$

We need to show that the above action is equal to the action $S_{\rm BPS}$ appearing in (3.41).

Let us consider first the hamiltonian term:

$$\begin{aligned}
\delta S_{\rm ham} &= \int d\tau \langle \frac{x_1^2+x_2^2}{2\hbar}\rangle \int \frac{dx_1 dx_2}{2\pi\hbar}\delta u(x_1,x_2,\tau) \\
&= \int d\tau \frac{\bar x_1^2 + \bar x_2^2}{2\hbar} \\
&= \int d\tau \frac{r^2}{2\hbar}
\end{aligned} \tag{4.5}$$

which matches with the hamiltonian term in (3.41). In the first step we have taken the integrand out of the cell δu since its size is small, in the second step we have used the fact that δu has area $2\pi\hbar$ and also equated the average values of x_1, x_2 with the coordinates of the centre of mass $\bar x_1, \bar x_2$ (see (3.39)) which satisfies $\bar x_1^2 + \bar x_2^2 = r^2$.

The analysis of the hamiltonian term for configuration (c) ((3.24),(3.25),(3.35)) is similar. It is interesting to note that in the special cases (3.18) and (3.23) the hamiltonian by convention measures the energy of the fluctuation δu together with that of a compensating fluctuation $\delta u'$ (see footnote 7) defined by adjusting the radius r_0 (this, again, corresponds to a choice of gauge for $C^{(4)}$ different from that in (3.29), (3.38)). Thus, e.g. the energy (3.16) includes the energy of the hole $-N\cos^2\theta$ as well as the energy $+N$ of the compensating outer circular strip $+\delta u'$, extending between r_0 and $r_0+\delta r_0$ such that the latter radius has an area $N+1$. In the generic case it is more natural to keep the two effects separate, which is possible to do in the semiclassical limit.

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The analysis of the kinetic term $\delta S_{\rm kin}$ is more complicated and is presented in appendix A. It is, however, somewhat simpler to match the equation of motion that follows from (4.4) with the equations of motion following from (3.41). The latter are

$$\dot{\bar{x}}_1 = \bar{x}_2\,, \qquad \dot{\bar{x}}_2 = -\bar{x}_1\,. \tag{4.6}$$

The equation of motion following from the action (4.4) can be read off from (4.3) and is given by

$$\dot{\delta u} - (x_1\partial_2 - x_2\partial_1)\delta u = 0\,. \tag{4.7}$$

Using the expression (3.4) for δu, one can show that (4.7) is satisfied to leading order in $\hbar$, provided (4.6) is valid.

4.2 Measure

The measure Du is defined as the group-invariant measure where u is parameterized as an orbit of some specific field configuration u_0 under the group of area-preserving diffeomorphisms (see [36, 37] and Section 6). The reference configuration u_0 satisfies $u_0^2 = u_0$ and $\int dx_1 dx_2\, u_0/(2\pi\hbar) = N$ so that the measure Du incorporates the two constraints (2.4) and (2.6).

When g acts on δu (see (3.4)), the action gets transmitted to the centres of mass of δu as a canonical transformation on $\bar{x}_1, \bar{x}_2$ (cf. (6.2)). The invariant measure under canonical transformations is the one already used in (3.41). We find, therefore, that the measures also agree.

5. Equivalence to Fermion path integral

Ref. [2] discussed the following path integral which represented a path integral for the phase space density $u(q,p,t)$ for free fermions moving in one dimension under a hamiltonian $h(q,p)$

$$\begin{aligned} Z_{\rm NC} &= \int [Du(q,p,t)]_{u_0} \exp[iS[u]] \\ S[u] &= \int \frac{dq\; dp}{2\pi\hbar}\hbar \int_{\tilde\Sigma} dt\; ds\; u(q,p,t,s) * \{\partial_t u, \partial_s u\}_{\rm MB} - \int_\Sigma dt\tilde{H} \\ \tilde{H} &= \int \frac{dq\; dp}{2\pi\hbar} u(q,p,t) * \frac{h(q,p)}{\hbar}\,. \end{aligned} \tag{5.1}$$

For free fermions moving in a harmonic oscillator potential

$$h(q,p) = \frac{p^2+q^2}{2}\,. \tag{5.2}$$

The star product in (5.1) is defined as

$$a * b(q,p) = \left[\exp\left(\frac{i\hbar}{2}\left(\partial_q\partial_{p'} - \partial_{q'}\partial_p\right)\right)\left(a(q,p)b(q',p')\right)\right]_{q'=q,p'=p}\,. \tag{5.3}$$

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The Moyal Bracket is defined as

$$\{a, b\}_{\rm MB} = \frac{a * b - b * a}{i\hbar}\,. \tag{5.4}$$

The measure Du is defined as the group-invariant measure under the symmetry group W_∞ of the fermion configuration space [2, 37]. The space of u's is the W_∞ orbit of a reference configuration u_0 which we can take to be the expectation value of the Wigner phase space distribution (5.9) in the Filled Fermi sea. The measure incorporates the constraint

$$u * u = u \tag{5.5}$$

and

$$\int \frac{dqdp}{2\pi\hbar} u = N\,. \tag{5.6}$$

The operator definition of the Wigner distribution $\hat{u}(q, p, t)$ is given in (5.9).

The equation of motion following from this path integral is

$$\begin{aligned}\partial_t u(q, p, t) &= \{h(q, p), u(q, p, t)\}_{\rm MB} \\ &= \{h(q, p), u(q, p, t)\}_{\rm PB} \\ &= (q\partial_p - p\partial_q)u(q, p, t)\end{aligned} \tag{5.7}$$

$h(q, p)$ is the single particle hamiltonian appearing in (5.1). The second step follows for any quadratic hamiltonian. For the $c = 1$ matrix model, one takes $h = (p^2 - q^2)/2$, but the analysis in [2] is true for any hamiltonian and in particular for $h = (p^2 + q^2)/2$. The third line follows from this latter hamiltonian. Although the equation of motion (5.7) coincides with its classical limit (4.3), the finite $\hbar$ dynamics differs significantly from its classical limit because the constraint (5.5) involves star products, involving fuzzy solutions for u [37, 38], unlike the constraint (2.4) whose solutions are characteristic functions (2.5). This is discussed further in the next two sections.

In [2] it was shown that (5.1) is exactly equal to a path integral for N free fermions moving in a simple harmonic oscillator potential, defined as follows:

$$\begin{aligned}Z_{\rm NC} = Z_F &= \int D[\Psi]_{|F_0\rangle} \exp\left[i\frac{S_F}{\hbar}\right] \\ S_F &= \int dt\, dx\, [\Psi^\dagger(x, t)(i\hbar\partial_t - h(x, \partial_x))\Psi(x, t)] \\ h &= \frac{1}{2}\left(-\hbar^2\frac{\partial^2}{\partial x^2} + x^2\right)\,.\end{aligned} \tag{5.8}$$

Here $\Psi(x, t)$, $\Psi^\dagger(x, t)$ are the second quantized annihilation and creation operators (respectively) for the fermions. The subscript $|F_0\rangle$ in the measure implies that the functional integral is over states obtained from the reference Fock space state $|F_0\rangle$ under W_∞ transformations. These in fact span all states with the same fermion number as $|F_0\rangle$, which we take to be N.

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Wigner phase space distribution. The Wigner phase space distribution $u(q,p,t)$ which appears in (5.1) as a path integral variable, can be defined as an operator (second quantized, see, e.g. [37, 38]) as follows:

$$\hat{u}(q,p,t) = \int d\eta \Psi^\dagger \left(q + \frac{\eta}{2}, t\right) \Psi(q - \eta/2, t) \exp\left[i\frac{p}{\hbar}\eta\right]. \tag{5.9}$$

Salient properties of this quantity as well those of its expectation values in various states have been listed in [37, 38, 36].

The correspondence. It is clear that (4.1) is simply the $\hbar \to 0$ limit of (5.1), provided one identifies $u(x_1, x_2)$ of Section 4 with $u(q,p)$ of this section. This is the advertised transformation of configuration space into phase space. The constraints (2.4) and (2.6) also follow from (5.5) and (5.6). Note that the equation $u * u = u$ reduces to $u^2 = u$ in the semiclassical limit, a fact which has been extensively exploited in [37, 38, 36].

Hence the collective coordinate quantization of LLM geometries gives rise to the $\hbar \to 0$ limit of free fermions in a harmonic oscillator potential. This is of course what we expect from the AdS/CFT correspondence [1], but we arrived at this result here starting from D-branes in supergravity. How to elevate this result to finite $\hbar$ remains an interesting issue. Some possible subtleties are mentioned in the next section. In the next section we also briefly discuss a more direct derivation of the semiclassical correspondence from supergravity using Kirillov's symplectic form.

6. Remarks on collective coordinate method with BPS constraint

In this section we will briefly discuss a first principles approach to the collective coordinate quantization of half-BPS geometries without using the D3 brane actions.

We begin by noting that the group G of time-independent area-preserving diffeomorphisms (SDiff) is a symmetry of the constraints (2.6) and (2.4), as well as of the equations of motion of the type IIB theory (since the geometries corresponding to various u's all satisfy IIB equations of motion). The Lie Algebra $\bar{G}$ is the algebra of symplectic vector fields. Thus, elements $g = 1 + X_f$ near identity of G, act on a function $u(x_1, x_2)$ as

$$\begin{aligned} u \to u^g &= u + X_f.u = u + \{f, u\}_{\mathrm{PB}} \\ X_f &= \epsilon_{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}. \end{aligned} \tag{6.1}$$

This action can also be regarded as induced by the motion of points on the plane under a hamiltonian f:

$$\begin{aligned} u^g(x) &= u(x^{g^{-1}}), \qquad \text{where} \\ (x_1, x_2)^g &\equiv \left(x_1 + \frac{\partial f}{\partial x_2}, x_2 - \frac{\partial f}{\partial x_1}\right). \end{aligned} \tag{6.2}$$

Finite group elements $g \in$ SDiff can be dealt with by exponentiation. Now, since the function u completely determines the supergravity fields (collectively denoted below as Φ): $\Phi = \Phi[u]$, the group G of area-preserving diffeomorphisms has a natural action on

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supergravity fields:

$$\Phi^g = \Phi[u^g]\,. \tag{6.3}$$

The choice of any given function u_0, and the corresponding Φ_0 breaks the symmetry $G \to H$, where H denotes the subgroup generated by functions which have zero Poisson bracket with u_0.

The collective coordinate method [39, 40] consists of making a change of variable $\Phi(t) \to \{g(t), \tilde\Phi(t) \equiv \Phi^{g(t)}(t)\}$, where $\tilde\Phi(t)$ represents motion in the body-fixed frame which is over and above the collective motion. The dynamics of the collective coordinate is obtained by implementing the change of variable in the field theory functional integral.

A first principles derivation of the collective coordinate action (without using the identification with D3 branes) would involve implementing the above procedure in the case of IIB supergravity. We will not attempt to do this here in detail, but give a brief discussion:

1. Since the IIB Lagrangian is second order in time derivatives, the low energy action for $g(t)$, is expected to be quadratic in $\dot g$ (before implementing the BPS condition). This corresponds, for example, to (3.11), (3.30) or (3.40), which are second order in time at low velocities. The phase space of the collective degree of freedom $g(t)$ at this stage involves $g(t)$ as well as $\pi_g(t)$ where $\pi_g(t)$ is the "momentum" for $g(t)$.

2. In case of the D3 brane dynamics one can explicitly see how (3.17) changes to (3.18) with the imposition of the BPS constraint (3.14). One would similarly expect that, if one implements the change of variable $\Phi \to \{g(t), \tilde\Phi\}$ in the IIB functional integral "in the presence of a BPS constraint", the dynamics of the collective variable $g(t)$ will be described by a first-order action and $g(t)$'s themselves would become a phase space. The most natural such an action on a G-orbit of a configuration u_0 is given by Kirillov's method of coadjoint orbits [37, 36, 41] (see, e.g. [36, eq. (68)])

$$S_{\rm BPS} = \int dt \langle X_t, u_0\rangle - \int dt \langle g^{-1} X_h g, u_0\rangle\,, \tag{6.4}$$

where $\langle X_f, u_0\rangle \equiv \int \frac{dx_1 dx_2}{2\pi\hbar}(f(x_1,x_2)u_0(x_1,x_2))$. The notation X_t denotes the Lie algebra element $g^{-1}\dot g$ and $X_h \equiv g^{-1}hg$ denotes the g-transported Lie algebra element corresponding to the hamiltonian $h = (x_1^2 + x_2^2)/2$. This action exactly coincides with (4.1) [37, 36]. Indeed the measure also coincides with the measure of (4.1). As a matter of fact we initially arrived at the action (4.1) by considering the Kirillov action [42]; from this viewpoint the D3-brane method can be viewed as additional evidence in support of the Kirillov action.

3. Evaluation of a functional integral "in the presence of a BPS constraint" involves insertion of an appropriate projection operator. It is possible that the resulting functional integral is an index, as in [43, 44], which are natural tools for counting geometries satisfying a specific number of supersymmetries.

4. If all $\Phi[u]$'s can be generated by the collective motion $\Phi[u^g]$, clearly the other degrees of freedom $\tilde\Phi$ are to be omitted from the functional integral under the half-BPS

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constraint. In this sense, the collective coordinate functional integral would appear to be the entire supergravity functional integral when subjected to the half-BPS constraint (see the next point, however).

5. There is an important subtlety regarding the number of connected components of the u-configurations. Although SDiff acts on the LLM geometries, it is not clear how it can change the number of connected components of a given u-configuration. Of course, under the W_∞ group mentioned in section 5, such a transformation can happen (a fuzzy droplet can split into two fuzzy droplets). However, W_∞ is the symmetry group of the equation $u * u = u$ and is not naturally associated with the LLM constraint $u^2 = u$. It remains unclear to us at the moment how to describe the entire space of LLM geometries as the orbit of a given configuration under a certain group G.

7. Conclusion

In this paper we considered collective coordinate quantization of LLM geometries identifying the function $z(x_1, x_2, 0) \equiv 1/2 - u(x_1, x_2)$ of [1] as the collective coordinate. The explicit form of the collective coordinate action (and measure) is derived by identifying the collective degree of freedom as that of a D3 brane coupled to an arbitrary LLM geometry. The D3 brane functional integral, subject to the BPS constraint, can be written directly in terms of the u-variable. We show that the resulting functional integral is the $\hbar \to 0$ limit of a functional integral describing free fermions in a harmonic oscillator potential. We discuss a first principles approach towards derivation of the u-integral using the general method of collective coordinates subject to a BPS constraint.

We note a few important points:

1. We find that supergravity configuration space becomes a phase space (hence noncommutative, with a noncommutativity parameter given by a certain $\hbar$), when constrained to configurations preserving a certain number of supersymmetries. Although we found this phenomenon in a specific case here (half-BPS IIB supergravity solutions with $\mathrm{O}(4) \times \mathrm{O}(4)$ symmetry), it is clear that this phenomenon should be generic. In particular the appearance of a first order action, discussed in section 6, is related to the fact that the BPS equations are first order. The formalism of phase space path integrals employed in this paper makes it rather apparent how a configuration space path integral with second order action becomes a phase space path integral with first order action under the imposition of the BPS constraint. It appears to be possible, using this, to count supersymmetric configurations within low energy field theories including supergravity. This observation clearly has implications for counting entropy of supersymmetric black holes and other related configurations.

2. As we mentioned in the previous section, functional integrals preserving a certain number of supersymmetries have earlier been treated in, for example, [43, 44], where the partition function is a 'twisted' one involving insertion of operators related to

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$(-1)^F$. It would be interesting to see if this is the case for half-BPS supergravity solutions treated in this paper. One would imagine defining such path integrals in terms of projection operators in the Hilbert space enforcing the supersymmetry conditions; it is of interest to explore the connection between this definition and the 'twisted' partition function mentioned above. Another related way of understanding "BPS functional integrals" would be to use topological twisting so that the relevant supersymmetry operators become BRST operators and the desired path integral becomes the normal path integral in the topological theory.

3. It is entirely possible, as in the context of the $c = 1$ matrix models, that the semi-classical collective excitation approach misses important subtle points of the fermion theory. In the case of $c = 1$ this was discussed in great detail in [45, 46, 37, 38, 47, 48]. One important effect missed by classical collective excitations (corresponding to the massless 'tachyons') is the unstable D0 brane of the two-dimensional string theory [49, 50] (this viewpoint is explained in [51]). In the present case, the semiclassical collective excitations consist of ripples (corresponding to gravitons, see appendix B) as well as D3 branes (roughly analogous to the tachyons and D0 branes, respectively, of two dimensional string theory). However, we might discover other important effects related to the non-perturbative description (5.1) possibly missed by the semiclassical treatment of the collective excitations.

4. We have used D3 branes coupled to LLM geometries to find noncommutative dynamics in the configuration space. It is interesting to note that in the limit of LLM geometries which describes D3 branes in the Coulomb branch [1], the value of $\hbar$ scales to zero causing the noncommutativity to disappear, as one would expect.

5. As seen in section 5, the phase space density action obtained from the fermion theory has an additional degree of noncommutativity reflected in the appearance of star products, over and above the noncommutativity mentioned in points (1) and (4). The latter is already evident in the semiclassical limit itself where the Moyal brackets get reduced to Poisson brackets and reflects a phase space structure of the classical configuration space. Clearly the former is related to the issue of finite $\hbar$ correspondence between the half-BPS geometries and the fermion theory. Of particular importance is whether the generalization to the constraint $u * u = u$ (instead of $u^2 = u$) allows some insight into g_{st} effects in string theory. Some aspects of the effect of finite g_s have been discussed at the end of the previous section.

6. A specific subleading $1/N$ correction briefly mentioned in this paper is the effect of the compensating fluctuations $\delta u'$ (see footnote 7). This effect is indeed calculable in the right hand side of (3.3) for various choices of $\delta u'$ and it is an interesting question whether the corresponding modification in the left hand side arises correctly by taking into account interaction between δu and $\delta u'$ coming from the star product structure of $S[u]$.

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7. Most of this paper dealt with collective excitations identified as D3 branes. We discuss gravitons briefly in appendix B; it would be interesting to quantitatively reproduce the graviton fluctuations from our collective action.

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A. Phase space density action for a single cell

In this appendix we will evaluate $\delta S_{\rm kin}$ appearing in (4.4), with δu as in (3.4). For simplicity of notation, we will denote

$$x_1 = q\,, \qquad x_2 = p\,. \tag{A.1}$$

Let us define

$$q^{\pm} = \pm\bar{q} + \frac{\epsilon}{2} \mp q\,, \qquad p^{\pm} = \pm\bar{p} + \frac{\epsilon}{2} \mp p\,. \tag{A.2}$$

Then

$$\delta u(q,p) = \theta(q^+)\theta(q^-)\theta(p^+)\theta(p^-)\,. \tag{A.3}$$

It is easy to calculate

$$\begin{aligned}\dot{\delta u} &= \dot{\bar{p}}\left[\delta(p^+)\theta(p^-) - \theta(p^+)\delta(p^-)\right]\theta(q^+)\theta(q^-) + \\ &\quad + \dot{\bar{q}}\left[\delta(q^+)\theta(q^-) - \theta(q^+)\delta(q^-)\right]\theta(p^+)\theta(p^-)\end{aligned} \tag{A.4}$$

and

$$\begin{aligned}\delta u' &= \bar{p}'\left[\delta(p^+)\theta(p^-) - \theta(p^+)\delta(p^-)\right]\theta(q^+)\theta(q^-) + \\ &\quad + \bar{q}'\left[\delta(q^+)\theta(q^-) - \theta(q^+)\delta(q^-)\right]\theta(p^+)\theta(p^-)\,.\end{aligned} \tag{A.5}$$

We define the Poisson bracket

$$\{f,g\}_{\rm PB} = \partial_q f \partial_p g - \partial_p f \partial_q g\,. \tag{A.6}$$

We get, after some simplification,

$$\begin{aligned}\delta S_{\rm kin} &= \int d\tau ds \int \frac{dqdp}{2\pi\hbar}\hbar\delta u\{\dot{\delta u}, \delta u'\}_{\rm PB} \\ &= \int \frac{3dp\ dq}{8\pi}\{\delta^2(q^+) + \delta^2(q^-)\}\{\delta^2(p^+) + \delta^2(p^-)\}\left[\int d\tau ds\ (\dot{\bar{q}}\bar{p}' - \dot{\bar{p}}\bar{q}')\right] \\ &= A\int d\tau\left(-\frac{r^2}{2\hbar}\dot{\tilde{\phi}}\right) \\ A &= \frac{3\hbar}{\pi}\delta_q(0)\delta_p(0)\,.\end{aligned} \tag{A.7}$$

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In the last line we have used eqs. (4.2) and (A.1) and the equality

$$\dot{q}\bar{p}' - \dot{p}\bar{q}' = \partial_s(\bar{p}\dot{q} - \bar{q}\dot{p}) = \partial_s(-r^2\dot{\tilde{\phi}})\,. \tag{A.8}$$

Thus $\delta S_{\rm kin}$ appearing in (4.4) agrees with the corresponding term in (3.41) apart from a proportionality constant A.

Let us discuss the constant A. In the last line of (A.7) $\delta_q(0)$ denotes $\delta(x_1 - x_1)$, similarly $\delta_p(0)$ denotes $\delta(x_2 - x_2)$. Clearly we need a regularization. It is natural to choose $\delta_q(0) = \delta_p(0) = a/\sqrt{\hbar}$. We get $A = 1$ if $a^2 = \pi/3$. We do not believe that this regularization has a particular significance since the agreement at the level of the equation of motion, between (4.6) and (4.7), does not use any such regularization. In other words, the equation of motion (4.7), which can be derived from (4.6), can be used to fix the relative coefficients between $\delta S_{\rm kin}$ and $\delta S_{\rm ham}$ in (4.4), thus determining $A = 1$ in (A.7). Such a method proves the desired result without the use of a regularization.

B. Gravitons

So far in this paper we have primarily considered collective motions identified as D3 branes. We found that (see (3.30)) the $\hbar$ of the collective action naturally corresponds to the D3-brane tension:

$$\frac{1}{2\hbar} = T_3\omega_3\,. \tag{B.1}$$

This raises a puzzle about other collective motions such as gravitons. Suppose we consider an equation analogous to (3.3), where the δu fluctuation corresponds to a ripple (see footnote 4) and the brane refers now to a fundamental string. Since the left hand side of (3.3) continues to have a prefactor $1/\hbar$ (see, e.g. (4.4), (4.5)), while the fundamental string tension does not involve $1/g_s$, we apparently have a puzzle here.

The resolution comes from the fact that δu now describes "ripples" which are fluctuations extending from the original droplet(s) by distances $\mathrm{O}(\sqrt{\hbar})$. Because of this, as we will show below, the collective action evaluates to $\mathrm{O}(g_s)$ which cancels the $1/g_s$, reproducing the fundamental string tension so far as g_s-counting is concerned.

The simplest parameterization [52, 45] for the ripples is as in figure 2. For simplicity we have considered the unperturbed droplet to correspond to $AdS_5 \times S^5$, but similar arguments can be made with respect to ripples traveling in other backgrounds.

The precise form of $u(x_1, x_2)$ is

$$u(x_1, x_2) = \theta([p^+(x_1) - x_2][x_2 - p^-(x_1)])\,, \tag{B.2}$$

where $p^{\pm}(x_1)$ are to be chosen consistent with (2.6). The fact that the amplitude of the fluctuations $\sim \mathrm{O}(\sqrt{\hbar})$ implies $\delta p^+, \delta p^- \sim \mathrm{O}(\sqrt{\hbar})$, where $\delta p^{\pm} = p^{\pm}(x_1) \mp p_0^{\pm}(x_1)$. $p_0^{\pm}(x_1)$ denote the unperturbed profile. Following steps similar to [52, 45] the action δS for the fluctuation turns out to be quadratic in $\delta p^+, \delta p^-$ and hence $\sim \mathrm{O}(\hbar) \propto \mathrm{O}(g_s)$. Thus, g_s cancels from the left hand side of (3.3) for ripples, consistent with their interpretation as fundamental string modes.

We hope to come back to a quantitative derivation of the action (as well as path integral) for gravitons from the collective coordinate path integral (4.1).

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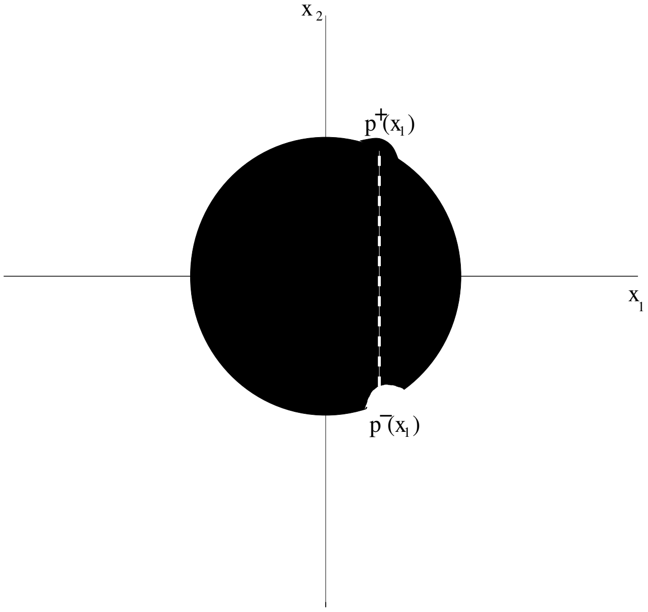

Figure 2: The fluctuations $p^+(x_1)$ and $p^-(x_1)$ extend from the original droplet to distances $\mathrm{O}(\sqrt{\hbar})$. The lagrangian for these fluctuations evaluates to $\mathrm{O}(g_s)$. This cancels the prefactor $1/g_s$ sitting outside the collective action (4.4), consistent with fundamental string tension which is independent of g_s.

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